Buckling of Short Cylindrical Shells under Axial Compression

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Approximate closed form solutions for the buckling of short circular cylindrical shells under uniform and nonuniform axial compression are obtained. The analysis is based on Donnell's equations using a Galerkin procedure to obtain bounds on the true eigenvalues. The boundary conditions considered are the four possibilities of simple support. Good agreement is shown with previous numerical results. The Koiter mode of instability is reconsidered through the examination of a simple model. The question as to its practical significance is raised.

Nomenclature

C_n = const
E = modulus of elasticity
K^2 = $[3(1-\nu^2)]^{1/2}R/h$
L = shell length
$M_x, N_x, N_{x\phi}$ = stress resultants
N_{xo} = prebuckling axial stress resultant
R = shell radius
Z_n = defined in Eq. (6)
h = shell thickness
k = integer, number of longitudinal half waves
m, n = integers, number of circumferential waves
p = integer, index
s_{p_2} = Fourier coefficients, see Eq. (24)
t_n^2 = $n^2/2K^2$
u^*, v^*, w^* = displacements, see Fig. 1
u, v, w = nondimensional displacements, $u = u^*/R$, $v = v^*/R$, $w = w^*/R$
u_n, v_n, w_n = defined in Eqs. (3a–3c)
x^*, z^*, ϕ = coordinates, see Fig. 1
x = nondimensional coordinates $x = x^*/R$
β = $\pi R/L$
δ_{mn} = Kronecker delta with $\delta_{oo} = 2$
μ = $\rho^*/\bar{\rho}$
ν = Poisson's ratio
ρ = true eigenvalue for uniform compression
ρ_n = defined in Eq. (5)
ρ_{NU} = true eigenvalue for nonuniform axial compression
$\bar{\rho}$ = defined in Eqs. (30a) and (30b)
ρ^* = approximate eigenvalue Eq. (27)
σ_{mn} = stress distribution matrix, see Eq. (29)
λ = $[1/(2)^{1/2}][3(1-\nu^2)]^{1/4}L/(Rh)^{1/2}$
Φ_n = defined in Eqs. (11a, 11b, 15a, and 15b)

Introduction

THE buckling problem of the circular cylindrical shell under uniform axial compression with various boundary conditions has been investigated in a number of papers.[1-4] In most studies with the few exceptions mentioned in Ref. 5, the solution is numerical. This is due to the fact that the buckling loads (eigenvalues) are usually obtained as the roots of a complicated transcendental equation.

The more realistic case of buckling under nonuniform axial compression has been considered by some authors for the classical (SS3) simply supported boundary condition,[6-8] and only recently[9] for the other possibilities of the simple support. The eigenvalues in this case are obtained by an iterative numerical solution for the characteristic roots of an infinite symmetric matrix.

The present work investigates analytically the buckling problem of short shells (defined here by $\lambda < 3$) under uniform and nonuniform axial compression with the four possibilities of the simply supported boundary conditions. Shells for which $\lambda < 3$ are frequently met in applications, for example, the "local" buckling of a cylindrical field between two stiffening rings. Although some numerical data[3,10-12] are available (for the case of uniform compression only), there is a need for further investigation of the problem, with emphasis on two points: a) how does the buckling behavior of a simply supported short shell approach that of a simply supported plate; and b) how is the critical stress, for short shells, influenced by nonuniform distribution of the axial load? These questions are investigated here, both quantitatively and qualitatively.

The first part of the paper deals with the uniform compression case. An approximate solution of Donnell's equations is obtained through the use of the Galerkin method. The analysis, although approximate and asymptotic in nature, is shown to be sufficiently accurate and yields close form formulas for the buckling load in the range $\lambda < 3$.

The case of nonuniform axial compression is considered in the second part of the paper. The solution for the displacements is represented as an infinite series in the approximate solutions for the case of uniform compression obtained before. Again the Galerkin method is applied and it is shown that for the limiting case $(h/R) \to 0$ the critical stress for any load distribution is closely bounded.

Part I. Uniform Compression

Formulation of the Problem and Method of Solution

The solution is based on Donnell's equation,[13] whose nondimensional form for uniform axial compression is

$$u_{,xx} + \frac{1-\nu}{2}u_{,\phi\phi} + \frac{1+\nu}{2}v_{,x\phi} - \nu w_{,x} = 0 \tag{1a}$$

$$\frac{1+\nu}{2}u_{,x\phi} + \frac{1-\nu}{2}v_{,xx} + v_{,\phi\phi} - w_{,\phi} = 0 \tag{1b}$$

$$\nabla^4 w + 12\left(\frac{R}{h}\right)^2 (w - \nu u_{,x} - v_{,\phi}) + 4K^2 \rho w_{,xx} = 0 \tag{1c}$$

Here u, v, w are the nondimensional displacements and ρ is the ratio (eigenvalue) between the buckling stress and the classical buckling stress $\{E/[3(1-\nu^2)]^{1/2}\}(h/R)$.

The four possibilities of the simply supported boundary conditions are known to be[1]

$$\left.\begin{aligned} &\text{Case SS1:} \quad w = M_x = N_x = N_{x\phi} = 0 && \text{(2a)}\\ &\text{Case SS2:} \quad w = M_x = u = N_{x\phi} = 0 && \text{(2b)}\\ &\text{Case SS3:} \quad w = M_x = N_x = v = 0 && \text{(2c)}\\ &\text{Case SS4:} \quad w = M_x = u = v = 0 && \text{(2d)} \end{aligned}\right\} \text{ at } x = 0, \frac{L}{R}$$

Received February 23, 1973. This paper is based, in part, on the first author's M.Sc. thesis.

Index categories: LV/M Structural Design (Including Loads); Structural Stability Analysis; Structural Static Analysis.

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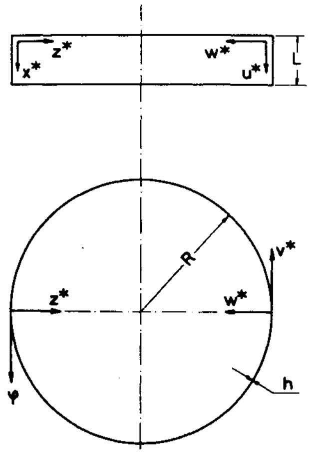

Fig. 1 Notation.

Since the shell is short, one may assume a double trigonometric expression for the radial displacement w for all the SS cases, and take the displacement field in the form

$$u_n = C_n[U_n(x)+a_n \cos k\beta x]\cos n\phi \tag{3a}$$

$$v_n = C_n[V_n(x)+b_n \sin k\beta x]\sin n\phi \tag{3b}$$

$$w_n = C_n \sin k\beta x \cos n\phi \tag{3c}$$

where a_n, b_n, C_n are constants and $U_n(x)$, $V_n(x)$ are to be regarded as correction functions depending on the boundary conditions. Note that w_n in Eq. (3c) satisfies the conditions $w = M_x = 0$ at $x = 0$, L/R. The index n emphasizes the dependence upon the number of circumferential waves n.

Substituting Eqs. (3a–3c) in the first two equilibrium Eqs. (1a) and (1b) it is found[9] that these equations are completely satisfied if a_n, b_n and $U_n(x)$, $V_n(x)$ are properly chosen.[14] The functions $U_n(x)$, $V_n(x)$ contain integration constants which are determined by the requirement that the displacements Eqs. (3a–3c) should satisfy the boundary conditions Eqs. (2a–2d).

After the displacements, Eqs. (3a–3c), have been determined up to an arbitrary constant C_n, the Galerkin method (which is, in this case, equivalent to the Rayleigh-Ritz procedure), is applied to Eq. (1c)

$$\int_o^{2\pi}\int_o^{L/R}\left[\nabla^4 w_n + 12\left(\frac{R}{h}\right)^2 (w_n - \nu u_{n,x} - v_{n,\phi}) + 4K^2\rho_n w_{n,xx}\right] w_n \, dx \, d\phi = 0 \tag{4}$$

Performing the integration, Eq. (4), an upper bound expression for ρ is obtained in the form[5]

$$\rho_n = \tfrac{1}{2}[Z_n + (1/Z_n)] + \Phi_n \tag{5}$$

where

$$Z_n = [n^2 + (k\beta)^2]^2/2K^2(k\beta)^2 \tag{6}$$

and Φ_n is an expression which depends on the boundary conditions.

In Eq. (5) the magnitude $\frac{1}{2}[Z_n + (1/Z_n)]$ is recognized to be the expression for ρ_n at the classical boundary condition SS3 (where of course $\Phi_n = 0$). The additional term Φ_n may be regarded as a correction depending on the boundary conditions. In what follows, compact and quite accurate formulas for the eigenvalues ρ_n in the range $\lambda < 3$ will be developed.

Case SS3

This is a well studied case[17] and only a short summary will be given with the present notation.

From Eq. (6) one obtains

$$Z_n = \frac{[(k\pi/2\lambda)^2 + t_n^{\,2}]^2}{(k\pi/2\lambda)^2} \tag{7}$$

where

$$t_n^{\,2} = n^2/2K^2 \tag{8}$$

Regarding $t_n^{\,2}$ as a continuous variable, the minimum value of ρ_n in Eq. (5) becomes:

$$(\rho_{SS3})_{min} = \frac{1}{2}\left[\left(\frac{k\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{k\pi}\right)^2\right], \quad \lambda \leqq \frac{\pi}{2}, \quad t_n^{\,2} = 0, \quad k = 1 \tag{9a}$$

$$(\rho_{SS3})_{min} = 1, \quad \lambda \geqq \frac{\pi}{2}, \quad t_n^{\,2} = \frac{k\pi}{2\lambda}\left(1 - \frac{k\pi}{2\lambda}\right), \quad (k = 1 \;\text{ if }\; \lambda < \pi) \tag{9b}$$

In the second part of the paper the dependence of ρ_n on $t_n^{\,2}$ for vanishing values of $t_n^{\,2}$ is needed. Hence, ρ_n is expanded in powers of $t_n^{\,2}$ (remembering that $\lambda < 3$ and $k \geqq 1$), resulting in

$$\rho_n = \frac{1}{2}\left[\left(\frac{k\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{k\pi}\right)^2\right] + 0(t_n^{\,2}) \tag{10}$$

Case SS4

In this case Φ_n is given by[9]

$$\Phi_n = \frac{1}{\lambda^2 K^2}\frac{4}{(1+\nu)^2}\frac{\left[\nu\left(\frac{k\pi}{2\lambda}\right)^2 - t_n^{\,2}\right]^2}{\left[\left(\frac{k\pi}{2\lambda}\right)^2 + t_n^{\,2}\right]^4} \times$$

$$\begin{cases} \dfrac{t_n \lambda \cosh^2 t_n \lambda}{\dfrac{3-\nu}{1+\nu}\sinh 2t_n\lambda - 2t_n\lambda} & \text{symmetric buckling} \quad k \text{ odd} \quad (11a) \\[2ex] \dfrac{t_n \lambda \sinh^2 t_n \lambda}{\dfrac{3-\nu}{1+\nu}\sinh 2t_n\lambda + 2t_n\lambda} & \text{antisymmetric buckling} \quad k \text{ even} \quad (11b) \end{cases}$$

The minimum value of ρ_n, given by Eq. (5) is considered first. A direct evaluation is very difficult due to the complicated expressions of Eqs. (11a) and (11b). The following procedure is therefore adopted: for any selection of $t_n^{\,2}$ and k, the resulting ρ_n is an upper bound for the true eigenvalue. Because of the similarity between the SS3 and SS4 cases,[1,5] the same values for $t_n^{\,2}$ and k are chosen to represent the upper bound. Also, from physical consideration[15] the exact eigenvalue for SS3 in Eqs. (9a) and (9b) is a lower bound for the true eigenvalue in the SS4 boundary condition (SS3 is a weaker condition than SS4). Noting that the present solution is an upper bound, one may conclude that the true eigenvalue in the SS4 boundary condition is bounded from above and below. Using the results of the previous section one gets

$$1 \leqq \frac{1}{2}\left[\left(\frac{\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{\pi}\right)^2\right] \leqq (\rho_{SS4})_{min} \leqq \frac{1}{2}\left[\left(\frac{\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{\pi}\right)^2\right] + \Phi_n(k = 1, t_n^{\,2} = 0), \quad \lambda \leqq \frac{\pi}{2} \tag{12a}$$

$$1 \leqq (\rho_{SS4})_{min} \leqq 1 + \Phi_n\left[k = 1, t_n^{\,2} = \frac{\pi}{2\lambda}\left(1 - \frac{\pi}{2\lambda}\right)\right], \quad \frac{\pi}{2} \leqq \lambda < 3 \tag{12b}$$

where Φ_n is given by Eqs. (11a) and (11b).

It is easy to see that the contributions of Φ_n in Eqs. (12a) and (12b) are negligible. For $\lambda \leqq \pi/2$

$$\Phi_n(k = 1, t_n^{\,2} = 0) = \frac{\nu^2}{1-\nu^2}\left(\frac{2}{\pi}\right)^4 \frac{\lambda^2}{K^2} \tag{13}$$

which is much smaller than unity. A similar result is obtained for $\pi/2 \leqq \lambda < 3$. The expressions for $(\rho_{SS4})_{min}$ are therefore given with very good accuracy by Eqs. (9a) and (9b).

It is noted that the preceding approximations should not be confused with those given in Ref. 5 for long shells.

Expansion of Φ_n in powers of $t_n^{\,2}$ and retention of only first terms yields

$$\Phi_n = \frac{\nu^2}{1-\nu^2}\left(\frac{2}{k\pi}\right)^4 \frac{\lambda^2}{K^2} \cdot \begin{cases} 1 + 0(t_n^{\,2}) & \text{symmetric buckling} \quad (14a) \\ 0(t_n^{\,2}) & \text{antisymmetric buckling} \quad (14b) \end{cases}$$

Comparing Eqs. (14a) and (14b) with Eq. (10), it is seen again that the contribution of Φ_n is negligible and the power expansion of ρ_n in the SS4 boundary condition is given almost exactly by Eq. (10).

Cases SS1–SS2

Here only symmetric buckling needs to be investigated since the antisymmetric mode yields in these cases much higher values for the buckling load for short shells.[1,16]

The results for Φ_n are the following:

Case SS1:

$$\Phi_n = -\frac{4}{\lambda^2}\frac{(k\pi/2\lambda)^4}{[(k\pi/2\lambda)^2+t_n^{\,2}]^4}\cdot\frac{t_n\lambda\cosh^2 t_n\lambda}{\sinh 2t_n\lambda+2t_n\lambda} \tag{15a}$$

Case SS2:

$$\Phi_n = -\frac{1}{2\lambda^2}\frac{1}{[(k\pi/2\lambda)^2+t_n^{\,2}]^2}\left[\frac{3(k\pi/2\lambda)^2-t_n^{\,2}}{(k\pi/2\lambda)^2+t_n^{\,2}}(t_n\lambda)\coth t_n\lambda + (t_n\lambda)^2(1-\coth^2 t_n\lambda)\right] \tag{15b}$$

Expansion of Φ_n in powers of $t_n^{\,2}$ yields for both conditions

$$\Phi_n = -(1/\lambda^2)(2\lambda/k\pi)^4 + 0(t_n^{\,2}) \tag{16}$$

Addition of Eq. (16) to Eq. (10) leads to the expression for ρ_n as $t_n^{\,2}\to 0$

$$\rho_n = \frac{1}{2}\left[\left(\frac{k\pi}{2\lambda}\right)^2+\left(\frac{2\lambda}{k\pi}\right)^2\right]-\frac{1}{\lambda^2}\left(\frac{2\lambda}{k\pi}\right)^4+0(t_n^{\,2}) \tag{17}$$

According to the results in Refs. 1 and 16, the minimum values of ρ_n in the SS1 and SS2 boundary conditions are obtained for vanishing values of $t_n^{\,2}$. In that case, the dominating part in Eq. (17) is minimized by $k = 1$ (for $\lambda < 3$). Hence an approximate upper bound for the lowest eigenvalues in the range $\lambda < 3$ is given by

$$(\rho_{SS1})_{min} = (\rho_{SS2})_{min} = \tfrac{1}{2}\{(\pi/2\lambda)^2+[1-(8/\pi^2)](2\lambda/\pi)^2\} \tag{18}$$

Naturally the question of the accuracy of expressions (9a, 9b, and 18) arises. A comparison of Eq. (18) with exact values obtained by numerical methods,[11] shows a very good agreement in the range $\lambda < 3$. More details are given in the next section. The fact that the same result holds for both SS1 and SS2 boundary conditions is due to their physical similarity which is of the same type as the similarity between the SS3 and SS4 boundary conditions.

Discussion of Results

Figure 2 describes the dependence of ρ_{min} on λ according to Eqs. (9a, 9b, and 18). Also are shown the exact numerical values obtained in Ref. 11. While the agreement between the two solutions in the SS3 and SS4 boundary conditions is expected [Eqs. (9a) and (9b) are the exact solution for case SS3 which is similar to case SS4], it is remarkable that Eq. (18) agrees very well with the numerical results of SS1 and SS2 in the range $\lambda < 3$.

Thus, Eqs. (9a, 9b, and 18) can be used to calculate the buckling load of short shells under axial compression with boundary conditions SS3–SS4 and SS1–SS2, respectively.

As $\lambda \to 0$ the results for all boundary conditions coincide with the critical stress of a long simply supported plate under axial compression. The critical stress of a long plate (with length b and width a) is given by Ref. 17

$$(\sigma_x)_{cr} = \frac{\pi^2 D}{b^2 h}\left(\frac{a}{b}+\frac{b}{a}\right)^2 \tag{19}$$

Taking $a = L$, $b = 2\pi R$ and dividing Eq. (19) by the classical buckling stress $(\sigma_x)_{cl} = E/K^2$ yields

$$(\rho)_{PLATE} = \frac{1}{2}\left(\frac{\pi}{2\lambda}\right)^2\left[1+\left(\frac{L}{2\pi R}\right)^2\right]^2 \approx \frac{1}{2}\left(\frac{\pi}{2\lambda}\right)^2 \tag{20}$$

Equation (20) is shown in Fig. 2.

The fact that long plate behavior (which is essentially a short column behavior) is approached by short shells, has already

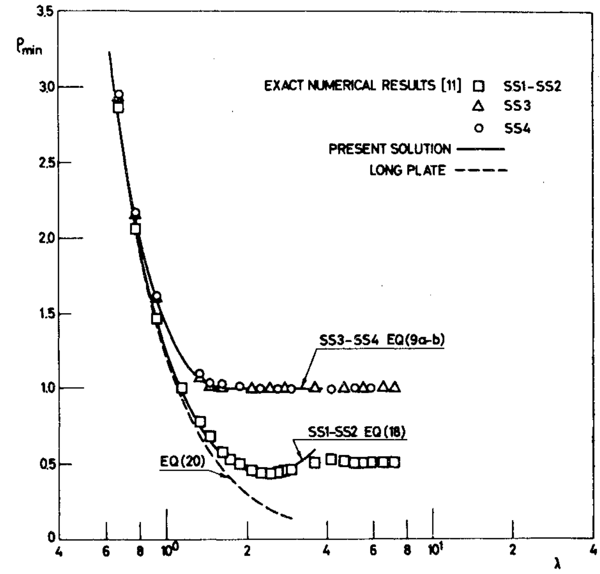

Fig. 2 Buckling of short shells under uniform axial compression

been pointed out by Flügge[18] for case SS3. A numerical solution for the other possibilities of the simply supported boundary conditions is given in Ref. 12 and again the long plate behavior is reached for short shells. The same conclusion has been obtained numerically also by Ohira[10] who used his own differential equations and a specific type of simply supported boundary conditions not included in the category SS1–SS4.

The behavior of short shells has recently become rather controversial. Koiter[19] was the first to suggest that for short simply supported shells the critical load is bounded from above by

$$\rho \leqq [\lambda^2/6(1-\nu^2)][1+2(1-\nu)(h/L)^2] \tag{21}$$

Koiter obtained this result by applying the Rayleigh-Ritz method to his own energy functional. In a later work Simmonds and Danielson[20] solved the equations developed by them in Ref. 21 and got, for a special relaxed simply supported boundary condition, that $\rho \to 0$ as $\lambda \to 0$. Their asymptotic solution is

$$\rho = \tfrac{1}{6}\lambda^2 \tag{22}$$

[For case SS3, however, a solution of the type (20) is given in Ref. 21.] A similar result,[22] has been obtained by solving exactly the differential equations derived from Koiter's modified energy functional.[23]

These results are of course in complete contrast with those obtained from Donnell's Eqs. (9a, 9b, and 18). A physical model of the special buckling mode associated with Eq. (22) has been offered in Ref. 20. This model is reconsidered in the Appendix of the present work and it is shown that if this buckling mode is to occur then the external compressive forces must be "glued" to the boundary—a situation which is not likely to appear in practice. See also in this respect Mangelsdorf's original study, a synopsis of which appears in Ref. 22.

Part II. Nonuniform Compression

Method of Solution

The formulation of the buckling problem under nonuniform axial compression remains unchanged except for the radial equilibrium equation, Eq. (1c), which takes the form

$$\nabla^4 w + 12\left(\frac{R}{h}\right)^2(w-\nu u_{,x}-v_{,\phi}) + 4K^2\rho_{NU}\left(\frac{1}{2}s_o+\sum_{p=1}^{\infty}s_p\cos p\phi\right)w_{,xx}=0 \tag{23}$$

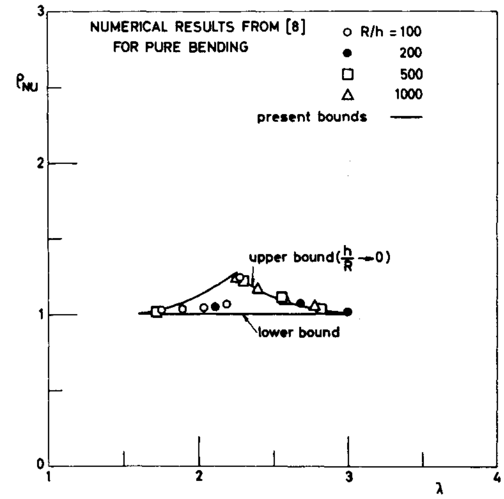

Fig. 3 Bounds for the buckling load under nonuniform compression, case SS3.

where the s_p are the Fourier coefficients of the series representation of the prebuckling axial stress resultant N_{xo}

$$N_{xo} = -\frac{Eh}{K^2}\rho_{\rm NU}\left(\frac{1}{2}s_o + \sum_{p=1}^{\infty} s_p \cos p\phi\right) \tag{24}$$

Here $\rho_{\rm NU}$ is the eigenvalue for nonuniform compression.

It is assumed without loss of generality that

$$\frac{1}{2}s_o + \sum_{p=1}^{\infty} s_p = 1 \tag{25}$$

Special cases of Eq. (24) are uniform compression ($s_o = 2$, $s_p = 0$, $p \neq 0$) already discussed in Pt. I, and pure bending ($s_1 = 1$, $s_p = 0$, $p \neq 1$).

In order to solve Eqs. (1a, 1b, and 23) the displacements are represented as an infinite series of the approximate solutions obtained in the case of uniform compression

$$u = \sum_{n=o}^{\infty} u_n = \sum_{n=o}^{\infty} C_n[U_n(x) + a_n \cos k\beta x]\cos n\phi \tag{26a}$$

$$v = \sum_{n=o}^{\infty} v_n = \sum_{n=o}^{\infty} C_n[V_n(x) + b_n \sin k\beta x]\sin n\phi \tag{26b}$$

$$w = \sum_{n=o}^{\infty} w_n = \sum_{n=o}^{\infty} C_n \sin k\beta x \cos n\phi \tag{26c}$$

Here a_n, b_n and $U_n(x)$, $V_n(x)$ are the same as before, see Eqs. (3a) and (3b), and the C_n are arbitrary constants.

The first two equilibrium equations, Eqs. (1a) and (1b), as well as the boundary conditions, are satisfied by Eqs. (26a–26c) and it remains to consider the third equation, Eq. (23). Applying the Galerkin method on this equation one obtains as a condition for a nontrivial solution the equation

$$\left|\int_o^{2\pi}\int_o^{L/R}\left[\nabla^4 w_n + 12\left(\frac{R}{h}\right)^2 (w_n - \nu u_{n,x} - v_{n,\phi}) + 4K^2\rho^*\left(\frac{1}{2}s_o + \sum_{p=1}^{\infty} s_p \cos p\phi\right) w_{n,xx}\right] w_m \, dx\, d\phi\right| = 0 \qquad m, n = 0, 1, \ldots \tag{27}$$

where the exact eigenvalue $\rho_{\rm NU}$ is replaced by ρ^* (obviously $\rho^* \geqq \rho_{\rm NU}$).

Performing the integration and using Eq. (4), which is an identity for w_n, yields

$$|\sigma_{mn} - (\rho_n/\rho^*)\delta_{mn}| = 0 \qquad m, n = 0, 1, \ldots \tag{28}$$

where δ_{mn} is the Kronecker delta (with $\delta_{oo} = 2$) and σ_{mn} is the stress distribution matrix

$$\sigma_{mn} = \tfrac{1}{2}(s_{|m-n|} + s_{m+n}) \tag{29}$$

The ρ_n are the approximate eigenvalues for the uniform compression case given by Eq. (5).

Thus, an upper bound for the critical load under nonuniform axial compression $\rho_{\rm NU}$ is given by the lowest ρ^* (largest $1/\rho^*$) satisfying Eq. (28).

This method of solution is an approximate version of a more general method suggested in Ref. 24. According to that method the displacements during buckling under nonuniform axial compression are described as a linear combination of the buckling modes obtained for uniform axial compression, i.e., the exact solutions of Eqs. (1a–1c). Although Eqs. (3a–3c) are not the exact solutions, they approach the exact solutions for short shells and, as justified by Fig. 2, are very accurate for $\lambda < 3$. In case SS3 the solution is exact.

Solution of Eq. (28) for Very Thin Shells

As the shell becomes very thin $(h/R) \to 0$, and one may neglect the terms of $0(t_n^2)$ in Eqs. (10) and (17), leaving the expression for ρ_n independent of n

$$(\rho_n)_{\rm SS3\text{-}SS4} \approx \frac{1}{2}\left[\left(\frac{k\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{k\pi}\right)^2\right] = \bar{\rho}_{\rm SS3\text{-}SS4} \tag{30a}$$

$$(\rho_n)_{\rm SS1\text{-}SS2} \approx \frac{1}{2}\left[\left(\frac{k\pi}{2\lambda}\right)^2 + \left(\frac{2\lambda}{k\pi}\right)^2\right] - \frac{1}{\lambda^2}\left(\frac{2\lambda}{k\pi}\right)^4 = \bar{\rho}_{\rm SS1\text{-}SS2} \tag{30b}$$

Equation (28) is therefore reduced to the equation

$$|\tfrac{1}{2}(s_{|m-n|} + s_{m+n}) - (1/\mu)\delta_{mn}| = 0 \qquad m, n = 0, 1, \ldots \tag{31}$$

where μ stands for the ratio

$$\mu = \rho^*/\bar{\rho} \tag{32}$$

and $\bar{\rho}$ is given by Eqs. (30a) and (30b).

It can be shown,[24] that an upper bound for μ, of Eq. (31) is

$$\mu \leqq 1 \tag{33}$$

hence

$$\rho^* \leqq \bar{\rho} \tag{34}$$

In addition, it is known[25] from the Courant maximum-minimum principle[26] that a lower bound for $\rho_{\rm NU}$ is $\rho_{\rm min}$, the exact eigenvalue for uniform compression, Eqs. (1a–1c). Using the previous results, the lower bound can be replaced, for short shells, by Eqs. (9a, 9b, and 18). Minimizing the upper bound $\bar{\rho}$ with respect to k the following is finally obtained:

$$\left.\begin{matrix}\lambda \leqq \dfrac{\pi}{2}, \dfrac{1}{2}\left[\left(\dfrac{\pi}{2\lambda}\right)^2 + \left(\dfrac{2\lambda}{\pi}\right)^2\right] \\ \dfrac{\pi}{2} \leqq \lambda < 3, \ 1\end{matrix}\right\} \leqq (\rho_{\rm NU})_{\rm SS3\text{-}SS4} \leqq$$

$$\left\{\begin{matrix}\dfrac{1}{2}\left[\left(\dfrac{\pi}{2\lambda}\right)^2 + \left(\dfrac{2\lambda}{\pi}\right)^2\right], & \lambda \leqq \dfrac{\pi}{(2)^{1/2}} & (35a) \\ \dfrac{1}{2}\left[\left(\dfrac{\pi}{\lambda}\right)^2 + \left(\dfrac{\lambda}{\pi}\right)^2\right], & \dfrac{\pi}{(2)^{1/2}} \leqq \lambda < 3 & (35b)\end{matrix}\right.$$

$$\lambda < 3, \frac{1}{2}\left[\left(\frac{\pi}{2\lambda}\right)^2 + \left(1 - \frac{8}{\pi^2}\right)\left(\frac{2\lambda}{\pi}\right)^2\right] \leqq (\rho_{\rm NU})_{\rm SS1\text{-}SS2} \leqq \frac{1}{2}\left[\left(\frac{\pi}{2\lambda}\right)^2 + \left(1 - \frac{8}{\pi^2}\right)\left(\frac{2\lambda}{\pi}\right)^2\right], \quad \lambda < 3 \tag{36}$$

Hence for very thin short shells, the critical stress in cases SS1–SS2, and cases SS3–SS4 (the latter for $\lambda \leqq \pi/2$ only) is almost independent of the load distribution. In cases SS3–SS4 for $\pi/2 \leqq \lambda < 3$ the critical stress for nonuniform load distribution is bounded between two close quantities. It is the author's opinion that the true eigenvalue is closer to the upper bound. This opinion is supported by two facts: a) the upper bound has been obtained as an eigenvalue of an infinite matrix; b) numerical results in case SS3 for bending (Table 1 in Ref. 8) are very close to the upper bound, Fig. 3. Note that the ratio $(\rho_{\rm NU})_{\rm SS3\text{-}SS4}/(\rho_{\rm NU})_{\rm SS1\text{-}SS2}$ can reach the value of $\sim 3[\lambda \approx \pi/(2)^{1/2}]$ while for the usual shells it is approximately 2.

Conclusions

Closed form expressions, Eqs. (9a, 9b, and 18), for the buckling loads, of short ($\lambda < 3$) circular cylindrical shells, under uniform axial compression and the four possibilities of simple support, have been developed. Their accuracy is demonstrated in Fig. 2. The short plate behaviour is reached as $\lambda \to 0$.

Closed form bounds, Eqs. (35a, 35b, and 36), are given for the buckling loads, of short shells, under nonuniform axial compression, in the limiting case $h/R \to 0$. The two bounds coincide in cases SS1–SS2, and in cases SS3–SS4 (for $\lambda < \pi/2$), with the corresponding buckling loads under uniform axial compression. It appears, Fig. 3, that the results are valid also for finite thickness and that the upper bound in cases SS3–SS4, for $\pi/2 < \lambda < 3$, is more reliable. Numerical work is under process to verify this point.

Appendix: Simple Models for External Load Variations at Buckling

A thin plate-like structure (which may also be curved out of plane to form a cylinder) is subjected along its boundaries, $X = \pm L/2$, to a compressive force system P acting in the X direction (Fig. 4). Along the boundaries $Y = 0; b$ the conditions of the plate are $v^* = \tau_{xy} = 0$. The stability of the structure is to be investigated under these conditions.

For this purpose, incremental displacements u^*, v^* (compatible with the geometrical restraints) are superposed on the loaded state in the following form:

$$u^* = -f(Y)$$
$$v^* = Xf' = X\gamma \quad \gamma(Y=0) = \gamma(Y=b) = 0$$

where $\gamma = f'$ and the prime denotes differentiation with respect to Y. In this state of deformation, elements which are initially oriented in the X direction neither change their length nor distort but are rotated through an angle γ. Calculation of the incremental strains yields

$$\varepsilon_y = X\gamma'; \quad \varepsilon_x = \gamma_{xy} = 0$$

Hence, the incremental elastic potential for this state of deformation becomes:

$$\Delta U = \int \frac{E}{2}\varepsilon_y^2\, dV = \frac{Eh}{2}\int\int (X\gamma')^2\, dX\, dY = \frac{EhL^3}{24}\int_o^b (\gamma')^2\, dY$$

In calculating the corresponding (quadratic in the displacements) incremental work of the external forces, the manner by which these forces change during the incremental deformations has to be specified. These terms are usually of secondary importance but may become the only terms left when special states of deformation are postulated such that the (usually) major terms drop out.

Two particular variations in the external force behavior are studied here:

Case a—"Donnell-type" behavior: The stress intensity on horizontal planes ($\Delta\sigma_x$) does not change during the incremental deformations. It is convenient to sum the incremental work along strips parallel to the X direction (see Fig. 4) and it is easy to see that no additional work is done. For Donnell-type behavior the structure is therefore stable under this particular incremental displacement mode.

Case b—"Koiter-type" behavior: The external loads on material elements do not change, $(\sigma_x + \Delta\sigma_x)\, dS_1 = \sigma_x\, dS$. In this case the loads remain "glued" to the structure so that the incremental stresses are proportional to the boundary strains ε_y:

$$\Delta\sigma_x = (P/h)\varepsilon_y(X = \pm L/2)$$

It is convenient, in this case, to sum the external work only along material lines. Each pair of elementary loads does work of magnitude

$$d(\Delta W) = L(1 - \cos\gamma)P\, dY \approx (PL/2)\gamma^2\, dY$$

The incremental potential of the external loads is therefore

$$\Delta V = -\Delta W = -\frac{PL}{2}\int_o^b \gamma^2\, dY$$

For stability, the total incremental potential is equated to zero, leading to

$$P_{cr}{}^* = \frac{EhL^2}{12}\frac{\int_o^b (\gamma')^2\, dY}{\int_o^b \gamma^2\, dY}$$

For any assumed γ (compatible with boundary restraints), the resulting $P_{cr}{}^*$ is an upper bound to the true critical force.

In the special case of the cylindrical shell, taking

$$dY = R\, d\phi$$
$$\gamma = A \sin n\phi$$

The resulting critical load is:

$$P_{cr}{}^* = (Eh/12)(nL/R)^2$$
$$(\sigma_{cr})_{min} = (1/h)(P_{cr}{}^*)_{min} = (E/12)(L/R)^2$$

It thus appears that instability is a property of the "glued" system of loads, and the critical stress approaches zero as $(L/R) \to 0$.

Similar results may apparently be obtained by introducing additional cases of variation in external load application as long as these are conservative and requirements of over-all equilibrium are met. The question arises as to the practical significance of the special load variations. For example, in the result obtained above (for $n = 1$), the resultant of the applied forces shifts away from the center of the cylinder by an amount of $AL/4$ during the incremental deformation. This behavior does not correspond, in the opinion of the authors, to a practical structure or laboratory experiment since in these cases the loads remain coaxial during the buckling process.

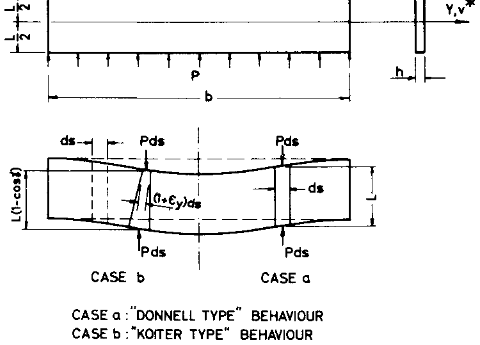

Fig. 4 Models for the behavior of short shells.

References

[1] Hoff, N. J. and Soong, T. C., "Buckling of Circular Cylindrical Shells in Axial Compression," *International Journal of Mechanical Sciences*, Vol. 7, 1964, pp. 489–520.

[2] Sobel, L. H., "Effects of Boundary Conditions on the Stability of Cylinders Subject to Lateral and Axial Pressures," *AIAA Journal*, Vol. 2, No. 8, Aug. 1964, pp. 1437–1440.

[3] Almroth, B. O., "Influence of Edge Conditions on the Stability of Axially Compressed Cylindrical Shells," *AIAA Journal*, Vol. 4, No. 1, Jan. 1966, pp. 134–140.

[4] Hoff, N. J., "The Perplexing Behaviour of Thin Circular Cylindrical Shells in Axial Compression," *Proceedings of the VIII Israel Anniversary Conference on Aviation and Astronautics, Israel Journal of Technology*, Vol. 4, No. 1, 1966, pp. 1–28.

[5] Durban, D. and Libai, A., "Buckling of a Circular Cylindrical Shell in Axial Compression and SS4 Boundary Conditions," *AIAA Journal*, Vol. 10, No. 7, July 1972, pp. 935–936.

[6] Abir, D. and Nardo, S. V., "Thermal Buckling of Circular Cylindrical Shells under Circumferential Temperature Gradients," *Journal of Aerospace Sciences*, Vol. 26, No. 12, 1958, pp. 803–808.

[7] Bijlaard, P. P. and Gallagher, R. H., "Elastic Instability of a Cylindrical Shell under Arbitrary Circumferential Variation of Axial

Stresses," *Journal of Aerospace Sciences,* Vol. 27, No. 11, 1959, pp. 854–858, 866.
[8] Seide, P. and Weingarten, V. I., "On the Buckling of Circular Cylindrical Shells under Pure Bending," *Journal of Applied Mechanics,* 1961, pp. 112–116.
[9] Durban, D., "On the Buckling of a Circular Cylindrical Shell under Non-Uniform Load and Various Boundary Conditions," M.Sc. thesis, 1971, Technion, Israel Institute of Technology, Haifa, Israel (in Hebrew).
[10] Ohira, H., "Local Buckling Theory of an Axially Compressed Circular Cylinder of Finite Length," *Proceedings of the 14th Japan National Congress for Applied Mechanics,* 1964.
[11] Yamaki, N. and Kodama, S., "Buckling of Circular Cylindrical Shells under Compression," Rept. 1, 1971, Institute of High Speed Mechanics, Tohoku Univ., pp. 99–123.
[12] Yamaki, N. and Kodama, S., "Buckling of Circular Cylindrical Shells under Compression," Rept. 2, 1972, Institute of High Speed Mechanics, Tohoku Univ., pp. 111–142.
[13] Donnell, L. H., "Stability of Thin Walled Tubes under Torsion," Rept. 479, 1933, NACA.
[14] Baruch, M. and Harari, O., "Buckling of Cylindrical Shells with Axial Surface Tractions," *Journal of Applied Mechanics,* 1969, pp. 350–352.
[15] Hoff, N. J. and Rehfield, L. W., "Buckling of Axially Compressed Circular Cylindrical Shells at Stresses Smaller than the Classical Critical Value," *Journal of Applied Mechanics,* 1965, pp. 542–546.
[16] Hoff, N. J., "Low Buckling Stresses of Axially Compressed Circular Cylindrical Shells of Finite Length," *Journal of Applied Mechanics,* 1965, pp. 533–541.
[17] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability,* 2nd ed., McGraw-Hill, New York, 1961.
[18] Flugge, W., "Die stabilität der Kreiszylinderschale," *Ingenieur-Archiv,* Vol. 3, 1932, pp. 463–506.
[19] Koiter, W. T., "General Equations of Elastic Stability for Thin Shells," *Proceedings of the Symposium on the Theory of Shells in Honor of L. H. Donnell,* Univ. of Houston, Houston, Texas, 1967, pp. 187–227.
[20] Simmonds, J. G. and Danielson, D. A., "New Results for the Buckling Loads of Axially Compressed Cylindrical Shells Subject to Relaxed Boundary Conditions," *Journal of Applied Mechanics,* 1970, pp. 93–100.
[21] Danielson, D. A. and Simmonds, J. G., "Accurate Buckling Equations for Arbitrary and Cylindrical Elastic Shells," *International Journal of Engineering Sciences,* Vol. 7, 1969, pp. 459–468.
[22] Mangelsdorf, C. P., "Koiter's Buckling Mode for Short Cylindrical Shells," *AIAA Journal,* Vol. 10, No. 1, Jan. 1972, pp. 13–14.
[23] Mangelsdorf, C. P., "Koiter's Modified Energy Functional for Circular Cylindrical Shells," *AIAA Journal,* Vol. 9, No. 10, Oct. 1971, pp. 2098–2099.
[24] Libai, A. and Durban, D., "A Method for Approximate Stability Analysis and its Application to Circular Cylindrical Shells under Circumferentially Varying Axial Loads," *Journal of Applied Mechanics,* 1973, pp. 971–976.
[25] Hill, D. W., "Buckling of a Thin Circular Cylindrical Shell Heated along an Axial Strip," SUDAER 88, 1959, Stanford Univ., Stanford, Calif.
[26] Courant, R. and Hilbert, O., *Methods of Mathematical Physics,* Vol. I, 1st ed., Interscience Publishers, New York, 1953, pp. 398–412.

JULY 1974 AIAA JOURNAL VOL. 12, NO. 7

Inelastic Buckling of a Deep Spherical Shell Subject to External Pressure

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This paper is concerned with the investigation of the inelastic buckling of a deep spherical shell subject to a uniformly distributed external pressure. The geometry of the shell is considered to be axisymmetrical while the shell thickness may vary as a function of the polar angle. The edge of the shell is supported elastically. The material of the shell is assumed to satisfy the generalized Ramberg-Osgood stress-strain relations and a power law of steady creep. The analysis is based on Sanders' nonlinear theory of thin shells expressed in an incremental form and Hill's theory of inelastic bifurcation. Computations are carried out by a numerical iterative procedure associated with a finite difference method. Solutions are sought for both the axisymmetrical inelastic buckling and the asymmetrical bifurcation.

I. Introduction

IN recent years, new developments in deep ocean exploration demand further refinement and sophistication in the design of submersibles. An important problem encountered in the design of submerged hulls is the selection of a suitable material with high-strength and low-density properties. In view of the nature of the material available and because of the limitations on structural weight, it is necessary that an accurate stress analysis of the hull and a reliable prediction of the collapse load be established.

When the submerged depth is large, the configuration of the hull is usually spherical. The spherical shell can provide a good structural layout from the viewpoint of stress analysis. Furthermore, it has the advantage of low drag during the motion of the hull. ONR's vehicle ALVIN is a typical example of the spherical hull. For this type of pressure hull, when it is submerged to a large depth, the deformation of the hull may become finite and the state of stress in the hull may reach the inelastic range. In a critical condition, structural failure may occur as a result of inelastic buckling.

Since Shanley first introduced the concept of inelastic bifurcation of a column under increasing axial compressive load,[1]

Received August 13, 1973; presented as Paper 74-34 at the AIAA 12th Aerospace Sciences Meeting, Washington, D.C., January 30–February 1, 1974. Research sponsored by the Office of Naval Research under Contract N00014-68-A-0152(NR 260-112/7-13-67).

Index categories: Structural Stability Analysis; Marine Vessel Design (Including Loads).

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